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THE SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AS FUNCTIONS OF THEIR INITIAL VALUES

By G. A. Bliss

The proof of the existence of solutions of the differential equation

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where the function f is expansible into a power-series in t and x, is included in most of the text books on differential equations. Methods have also been devised for proving the existence of an integral taking a given value ξ for $t = \tau$, when the function f satisfies much less stringent restrictions.* In many applications however the mere existence theorem is of little value unless accompanied by a proof of the continuity and differentiability of the solution considered as a function of the initial constants τ, ξ. As the literature dealing with these latter questions is somewhat scattered and the proofs complicated, it is proposed in the following pages to collect and simplify as far as possible the results of the different writers. Solutions are defined reaching from boundary to boundary of the region R over which the properties of the function f are assumed provided R is closed, and the method of Peano† for proving the existence and continuity of the first partial derivatives with respect to the initial constants has been extended by a simple device to the higher derivatives.

1. The existence of solutions. In order to make the existence proof suppose 1) that the function $f(t, x)$ in equation (1) is continuous in the interior of a region‡ R of the tx-plane, and 2) that a positive constant K' can be found for any finite closed region R' interior to R, such that

$$|f(t, x) - f(t, x')| \leqq K'|x - x'|. \tag{2}$$

* See for example, Osgood, *Monatshefte für Mathematik*, vol. 9, 1898. Only the continuity of $f(t, x)$ is assumed. For other references, see *Encyklopädie der mathematischen Wissenschaften*, II A 4a, especially pp. 195, 200.

† *Atti di Torino*, vol. 33, p. 9 ff.

‡ In the following pages the word region means always a point set with interior points. A closed region is one which contains all its limit points.

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where (t, x) and (t, x') are any two points for which the interval $[x, x']$ on the t-ordinate consists only of points in R'.*

Consider the series of approximation functions:†

$$(3)\qquad \begin{cases} x^{(1)} = \xi + \int_\tau^t f(t, \xi)\,dt \\ x^{(2)} = \xi + \int_\tau^t f(t, x^{(1)})\,dt \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x^{(n)} = \xi + \int_\tau^t f(t, x^{(n-1)})\,dt \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \end{cases}$$

where (τ, ξ) is a point interior to the region R through which it is desired to pass a solution of equation (1). Any one $x^{(n)}$ of these functions is surely well-defined on an interval

$$(4)\qquad |t - \tau| \leqq l$$

for which the preceding function $x^{(n-1)}$ is continuous and the points $(t, x^{(n-1)})$ all lie in the region R, and each function takes the initial value ξ when $t = \tau$. An interval (4) can be found by selecting a positive constant ρ so that all the points which satisfy the inequalities

$$(5)\qquad R_1: \qquad |t - \tau| \leqq \rho, \quad |x - \xi| \leqq \rho$$

are interior to the region R, and then taking l as the smaller of ρ and $\dfrac{\rho}{M_1}$, where M_1 is the maximum of the absolute value of f in the neighborhood (5) of the point (τ, ξ). For then it follows from the relations

$$|x^{(n)} - \xi| = \left|\int_\tau^t f(t, x^{(n-1)})\,dt\right| \leqq M_1 |t - \tau| \leqq \rho, \qquad (n = 1, 2, \cdots),$$

that all the points $(t, x^{(1)})$, $(t, x^{(2)})$, etc., are in the neighborhood (5).

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* This is the so-called Lipschitz condition (see *Encyclopädie der Math. Wiss.*, II A 4a, p. 194). It will certainly be satisfied if f has a continuous partial derivative $f_x = \dfrac{\partial f}{\partial x}$ in R. For then

$$f(t, x) - f(t, x') = (x - x')\, f_x(t, x + \theta(x - x')), \qquad 0 < \theta < 1$$

provided that the interval $[x, x']$ of the t-ordinate is all in R', and the maximum of $|f_x|$ in R' would be a constant k of the kind specified.

† Compare Picard, *Traité d'analyse*, vol. 2, p. 301.

The series

$$(6) \qquad x = \xi + (x^{(1)} - \xi) + (x^{(2)} - x^{(1)}) + \cdots$$

is uniformly convergent* for all values of t on the interval (4). Namely from (2),

$$(7) \qquad |x^{(n)} - x^{(n-1)}| \leqq \left| \int_\tau^t \{ f(t, x^{(n-1)}) - f(t, x^{(n-2)}) \} dt \right|$$

$$\leqq \left| \int_\tau^t k_1 |x^{(n-1)} - x^{(n-2)}| dt \right|, \qquad (n > 1),$$

where k_1 is the constant in (2) belonging to the neighborhood R_1. Let N be the maximum of $|f(t, \xi)|$ on the interval (4). Then

$$|x^{(1)} - \xi| \leqq N |t - \tau|,$$

and by successive application of the inequality (7),

$$|x^{(n)} - x^{(n-1)}| \leqq \frac{N}{k_1} \frac{k_1^n |t - \tau|^n}{n!}.$$

The absolute value of any term of the series (6) is therefore less than the corresponding term of the development of $\frac{N}{k_1} \{ e^{k_1 |t-\tau|} - 1 \}$, which is a series of positive terms converging uniformly for all values of t in the interval (4). From this the uniform convergence of (6) follows.

The continuous function $x(t)$ defined by the series (6) *on the interval* (4), *takes the value ξ for $t = \tau$ and satisfies the differential equation* (1). For since the region (5) is a closed one the function f is uniformly continuous,† *i. e.* for any positive ϵ a positive δ can be found such that

$$|f(t', x') - f(t, x)| < \epsilon,$$

where (t', x') and (t, x) are any two points of R_1 satisfying

$$|t' - t| < \delta, \quad |x' - x| < \delta.$$

Furthermore on account of the uniform convergence of the series (6) a positive integer m can be found such that if $n > m$,

$$|x^{(n-1)} - x| < \delta.$$

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* Here, as in what follows, the fundamental theorems concerning uniform convergence are assumed. See, *e. g.*, Jordan, *Cours d'analyse*, vol. 1, p. 310 ff.

† A function which is continuous in a finite closed region, is uniformly continuous; see Jordan, *loc. cit.*, p. 48.

Therefore also

$$\left|\int_\tau^t \{ f(t, x^{(n-1)}) - f(t, x) \} dt \right| < \epsilon l$$

for $n > m$, and

$$\lim_{n=\infty} \int_\tau^t f(t, x^{(n-1)}) dt = \int_\tau^t f(t, x) dt.$$

By letting n approach infinity on both sides of the equation (3), it follows that

(8) $$x = \xi + \int_\tau^t f(t, x) dt,$$

from which the statements made above are easily proved by putting $t = \tau$ and by differentiation.

A solution having been defined in the interval $[\tau, \tau + l]$, the same process can be applied again with the point $\bigl(\tau + l,\ x(\tau + l)\bigr)$ as initial point, provided that this point is still within R, and the solution can be extended to an interval $[\tau + l,\ \tau + l + l']$. If the region R is finite and closed the values of t which can be reached by such continuations must have an upper bound t_1, and the points $\bigl(t,\ x(t)\bigr)$ approach a definite limit point as t approaches t_1. For if t' and t'' both satisfy the inequality

$$0 < t_1 - t < \frac{\epsilon}{M},$$

where M is the maximum of $|f|$ in the region R, then equation (8) shows that

(9) $$|x(t') - x(t'')| \leqq M\,|t' - t''| < \epsilon,$$

and this is the necessary and sufficient condition that the function $x(t)$ has a unique limiting value for $t = t_1$.* The limit point $\bigl(t_1,\ x(t_1)\bigr)$ must be a boundary point of the region R. Otherwise it would be an interior point, and in that case t_1 would not be the upper bound of the t-values which could be reached by continuation.

---

*Suppose $t', t'', \ldots, t^{(n)}, \ldots$ have the limit t_1 and are less than t_1. By Jordan, *loc. cit.*, p 9, the corresponding values of x define a unique limit. But by (9) this is the limit which x approaches as t approaches t_1 over any set of values whatsoever.

On the other hand, if the region R is not finite and closed, it might happen that the solution could be extended over any t-interval, or if t_1 were finite perhaps the function $x(t)$ would have no unique limit at $t = t_1$. In this case also, however, if t_1 is finite, and if $x(t)$ approaches a unique limit as t approaches t_1, then it can be proved as above that the point $\big(t_1, x(t_1)\big)$ must be a boundary point of R.

The preceding discussion justifies then the following theorem:

Through any point (τ,ξ) *in the interior of the region* R *in which the function* f *is continuous and satisfies the Lipschitz condition* (2), *a solution of equation* (1), *denoted by*

$$x = \phi(t,\tau,\xi), \tag{10}$$

can be passed. The function ϕ *is continuous in* t *and defines points* (t,x) *in the interior of* R *for all values of* t *in an interval*

$$t_0 < t < t_1,$$

in which the value τ *is included.*

If the region R *is finite and closed, the interval* $[t_0,t_1]$ *is finite and can be chosen so that as* t *approaches* t_0 (*or* t_1), *the points* (t,x) *of the solution approach a unique limit point on the boundary of the region.*

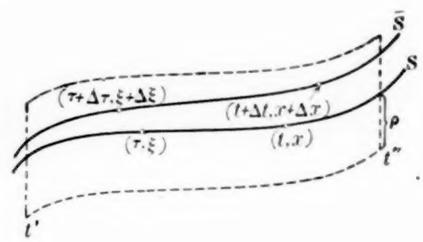
If R *is not finite and closed, the largest possible interval* $[t_0, t_1]$ *may be infinite, or if finite the solution may not have definite limit points at its extremities* t_0 *and* t_1. *In case* t_0 (*or* t_1) *is finite and the solution approaches a unique limit point, then as before the limit point must belong to the boundary of* R.

For some applications it is important to notice that *if* P *is a finite closed region entirely interior to* R, *a positive quantity* λ *can be determined, such that the solution through any point* (τ, ξ) *of* P *can be extended at least over the interval* $|t - \tau| \leqq \lambda$. This follows because a constant ρ can be chosen independent of the position of (τ, ξ) in P and so that the corresponding region (5) is entirely interior to R. If M is the maximum of $|f|$ in the region consisting of all the regions (5) about points of P, then according to the preceding discussion the smaller of ρ and $\dfrac{\rho}{M}$ will be a constant λ with the desired property.

2. Uniqueness of the solution.* Consider a solution $S\big(x = x(t)\big)$ of equation (1), defined and interior to the region R for all values of t in an

*Compare the methods used in §§2, 3, 4 with Peano, *loc. cit.*

interval $[t', t'']$. A positive constant ρ can be so selected that all points of the neighborhood R_2 of S indicated by the dotted lines in the figure, are also interior to R. Let $\bar{S}\Big(y = y(t)\Big)$ be another solution such that the value $t = \tau$,



and therefore all values in a certain interval T including τ, define points (t, y) interior to the region R. Then the difference $(x - y)$ satisfies the equation

$$\frac{d(x-y)}{dt} = f(t, x) - f(t, y),$$

and the forward derivative* of $|x - y|$ satisfies the inequality

$$(11) \qquad \left|\frac{d|x-y|}{dt}\right| = \left|\frac{d(x-y)}{dt}\right| \leqq k_2 |x - y|$$

on account of the condition (2), where k_2 is the value of k belonging to the region R_2. Consequently

$$\frac{d|x-y|e^{-k_2(t-\tau)}}{dt} \leqq 0, \quad \frac{d|x-y|e^{k_2(t-\tau)}}{dt} \geqq 0.$$

By using the former inequality for $t > \tau$ and the latter for $t < \tau$, it follows that

$$(12) \qquad |x - y| \leqq |\xi - \eta| e^{k_2|t-\tau|}$$

for all values of t on the interval T, where ξ and η are the values of x and y

*The forward derivative is used because at points where a function $u(t)$ vanishes, the derivative of $|u| = +\sqrt{u^2}$ is discontinuous. At such values of t, however, the *forward* derivative exists, and by the mean-value theorem is

$$\frac{d\sqrt{u^2}}{dt} = \lim_{h=0} \frac{1}{h}\sqrt{[hu'(t+\theta h)]^2} = +\sqrt{\left(\frac{du}{dt}\right)^2},$$

where h is supposed to approach zero from the positive side. At other points the first part of (11) is easily proved.

for $t = \tau$.* It follows, then, that if S and $\bar{S}$ intersect at any point $t = \tau$, $\xi = \eta$, the value of $|x - y|$ is less than or equal to zero, and x must be identically equal to y for all values of t in the common interval adjoining τ in which the two solutions are defined.

Only one solution of equation (1) *can pass through a given point* (τ, ξ) *interior to* R. *The set* (10) *includes, therefore, all the solutions in the region.*

3. Continuity of the solutions with respect to the initial values. With the help of the inequality (12) it can be shown that the solution (10) through the point (τ, ξ) can be surrounded by solutions corresponding to neighboring initial values $(\tau + \Delta\tau, \xi + \Delta\xi)$, and that these go over continuously into the former when $\Delta\tau$ and $\Delta\xi$ approach zero. Let S denote the solution (10), and suppose that an interval $[t', t'']$ and a neighborhood R_2 of the kind described in §2 have been found. $\Delta\tau$ and $\Delta\xi$ are to be so chosen that the point $(\tau + \Delta\tau, \xi + \Delta\xi)$ lies in R_2, and $\bar{S}$ will be used to denote the solution

$$x = \phi(t, \tau + \Delta\tau, \xi + \Delta\xi)$$

through that point. Then from equation (8)

$$|\phi(\tau + \Delta\tau, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(\tau + \Delta\tau, \tau, \xi)| = \left|\Delta\xi - \int_\tau^{\tau+\Delta\tau} f(t, \phi)\,dt\right| \leqq |\Delta\xi| + M_2|\Delta\tau|,$$

where M_2 is the maximum of $|f|$ in the neighborhood R_2. This gives the difference of the values of x for S and $\bar{S}$ at $t = \tau + \Delta\tau$, so that when τ is replaced by $\tau + \Delta\tau$, the inequality (12) becomes

$$(13) \qquad |\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)| \leqq \{|\Delta\xi| + M_2|\Delta\tau|\}e^{k_2|t-\tau-\Delta\tau|}.$$

Suppose now that $\Delta\tau$ and $\Delta\xi$ are still further restricted so that

$$(14) \qquad \{|\Delta\xi| + M_2|\Delta\tau|\}e^{k_2|t''-t'|} < \rho.$$

Then the solution $\bar{S}$ through $(\tau + \Delta\tau, \xi + \Delta\xi)$ can be continued over the whole

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* *If a function* $f(t)$ *is continuous and has a forward derivative* $f'(t) \leqq 0$ *on the interval from* t_0 *to* t_1 $(t_0 < t_1)$, *then* $f(t_0) \geqq f(t_1)$. For, the function

$$\phi(t) = f(t) + \beta t,$$

where β is any negative constant, has a negative forward derivative. If $\phi(t_0)$ were less than $\phi(t_1)$, the function ϕ would attain its minimum at some value t' less than t_1. At that point the forward derivative would necessarily be $\geqq 0$, contrary to the hypothesis. Consequently

$$\phi(t_0) - \phi(t_1) = f(t_0) - f(t_1) - \beta(t_1 - t_0) > 0$$

for every negative β, which proves the theorem stated for $f(t)$.

interval $t't''$. For by the results of §1, the process of continuation can be carried on until the boundary of R_2 is reached, and on account of the inequalities (13) and (14) the only points attained by such continuations between the ordinates $t = t'$ and $t = t''$ are in the interior of R_2. The ordinates $t = t'$ and $t = t''$ are therefore the only boundaries which the solution $\bar{S}$ can cross.

From these results it follows that if t, $t + \Delta t$ are any two values in the interval $[t', t'']$, then by applying (8) and (13),

$$\begin{aligned} |\Delta\phi| &= |\phi(t + \Delta t,\ \tau + \Delta\tau,\ \xi + \Delta\xi) - \phi(t, \tau, \xi)| \\ &\leqq |\phi(t + \Delta t,\ \tau + \Delta\tau,\ \xi + \Delta\xi) - \phi(t, \tau + \Delta\tau,\ \xi + \Delta\xi)| \\ &\qquad + |\phi(t, \tau + \Delta\tau,\ \xi + \Delta\xi) - \phi(t, \tau, \xi)| \\ &\leqq M_2 |\Delta t| + \{|\Delta\xi| + M_2|\Delta\tau|\} e^{k_2(t-\tau-\Delta\tau)}. \end{aligned}$$

Therefore the function (10) *is continuous for all values of t, τ, ξ defining points of the solution interior to the region R.* For, as has been indicated, an interval $[t', t'']$ including the values t and τ can always be selected, and $\Delta\tau$, $\Delta\xi$ restricted, so that the solutions $\bar{S}$ are defined on the interval and satisfy the last inequality.

4. Partial derivatives with respect to the constants of integration. In order to prove the existence of the partial derivatives $\dfrac{\partial\phi}{\partial\tau}$, $\dfrac{\partial\phi}{\partial\xi}$ of the function (10) at a point t, τ, ξ, the additional assumption is made that the derivative $f_x = \dfrac{\partial f}{\partial x}$ exists and is continuous in the interior of the region R. For the two solutions S, $\bar{S}$ of §3, it follows from equation (1) that

$$\frac{d\Delta\phi}{dt} = a\Delta\phi, \tag{15}$$

where

$$\Delta\phi = \phi(t,\ \tau + \Delta\tau,\ \xi + \Delta\xi) - \phi(t, \tau, \xi),$$

$$a = \frac{f(t, \phi + \Delta\phi) - f(t, \phi)}{\Delta\phi}.$$

In these expressions τ and ξ are considered fixed, for the moment, while t, $\Delta\tau$, $\Delta\xi$ are variable. The coefficient a is a continuous function of t, $\Delta\tau$, $\Delta\xi$ when t is on the interval $[t', t'']$, and $\Delta\tau$, $\Delta\xi$ satisfy (14).* This follows from §3 for any values of the arguments for which $\Delta\phi \neq 0$, and with the help of the expression

$$a = f_x(t, \phi + \theta\Delta\phi), \qquad 0 < \theta < 1,$$

*See a note by Hadamard, *Bull. de la société math.*, vol. 28 (1900), p. 64.

for any values for which $\Delta\phi = 0$, on account of the continuity of the derivative f_x. The general integral of equation (15) is therefore

$$\Delta\phi = ce^{\int_\tau^t a\,dt}.$$

When $\Delta\tau = 0$, the constant c has the value $\Delta\xi$, and it follows easily that the quotient $\dfrac{\Delta\phi}{\Delta\xi}$ has the limit

$$\frac{\partial\phi}{\partial\xi} = e^{\int_\tau^t f_x(t,\phi)dt}, * \tag{16}$$

which is a continuous function of t, τ, ξ on account of the continuity of ϕ and f_x. Similarly when $\Delta\xi = 0$,

$$c = \Delta\phi\,|^{t=\tau} = \int_{\tau+\Delta\tau}^{\tau} f(t, \phi + \Delta\phi)dt$$

$$= -\Delta\tau f\Big(\tau + \theta\Delta\tau,\ \phi(\tau + \theta\Delta\tau,\ \tau + \Delta\tau,\ \xi)\Big),\ 0 < \theta < 1,$$

by (8) and the mean value theorem for a definite integral. At the limit therefore

$$\frac{\partial\phi}{\partial\tau} = -f(\tau, \xi)e^{\int_\tau^t f_x(t,\phi)dt}. \tag{17}$$

Differentiation of equation (8) twice for t shows that it the function f has the derivative $\dfrac{\partial f}{\partial t}$ continuous in the interior of R, then the derivative $\dfrac{\partial^2\phi}{\partial t^2}$ exists and is also continuous. If furthermore all the second derivatives of f except perhaps $\dfrac{\partial^2 f}{\partial t^2}$ are continuous, the equations (16) and (17) can be differentiated, and ϕ will have all the derivatives of the second order. A simple induction leads to the theorem:

When the function $f(t, x)$ has all its derivatives of the $(n-1)^{st}$ order, and all of the n^{th} order except perhaps $\dfrac{\partial^n f}{\partial t^n}$, continuous in the interior of the region R, then the solution (10) has continuous derivatives of the n^{th} order with respect to t, τ, ξ at any set of values (t, τ, ξ) defining a point (t, x) of the solution interior to R.

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* See Bendixon, *Bull. de la société math.*, vol. 24 (1896), p. 220.

5. Complex numbers.* The results of the preceding sections can be readily extended to a *system* of differential equations by means of complex numbers and matrices. The necessary definitions follow:

1) x is a symbol for the set $(x_1, x_2, \cdots, x_r)$;

2) $\operatorname{mod} x = \sqrt{x_1^2 + x_2^2 + \cdots + x_r^2}$;

3) $x \gtreqless y$ if $x_i \gtreqless y_i \quad (i = 1, 2, \cdots r)$;

4) $x \pm y$ are the complex numbers $(x_1 \pm y_1, x_2 \pm y_2, \cdots, x_r \pm y_r)$;

5) xy is the sum $\sum_{i=1}^{r} x_i y_i$;

6) kx is the complex number $(kx_1, kx_2, \cdots, kx_r)$ if k is a simple constant;

7) if the elements of x are functions of t, then $\int x dt$ and $\dfrac{dx}{dt}$ are the complex numbers $\left(\int x_1 dt, \int x_2 dt, \cdots, \int x_r dt\right)$ and $\left(\dfrac{dx_1}{dt}, \dfrac{dx_2}{dt}, \cdots, \dfrac{dx_r}{dt}\right)$ respectively;

8) $|x|$ is the complex number $(|x_1|, |x_2|, \cdots, |x_r|)$;

9) A denotes a matrix of elements $a_{ij} \quad (i, j = 1, 2, \cdots, r)$;

10) Ax is the complex number of which the elements are $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ir}x_r \quad (i = 1, 2, \cdots, r)$;

11) $A = B$ if $a_{ij} = b_{ij} \quad (i, j = 1, 2, \cdots, r)$;

12) AB is the matrix C whose elements are $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} \quad (i, j = 1, 2, \cdots, r)$;

13) $\dfrac{dA}{dt}$ is the matrix whose elements are $\dfrac{da_{ij}}{dt} \quad (i, j = 1, 2, \cdots r)$;

14) $\dfrac{dx}{d\xi}$, when x and ξ are two complex numbers such that the elements of x are functions of the elements of ξ, is the matrix of derivatives $\dfrac{\partial x_i}{\partial \xi_j} \quad (i, j = 1, 2, \cdots r)$.

Besides these definitions the following three theorems are needed:

a) $\operatorname{mod}(x \pm y) \leqq \operatorname{mod} x + \operatorname{mod} y$;

*For the application of complex numbers to linear equations, see Peano, *Math. Annalen*, vol. 32 (1888), p. 450. In the article by Peano cited previously, they are used without explanation for any system of equations in the so-called normal form.

b) if the elements of x are functions of t having continuous derivatives, then mod x has a forward derivative and

$$\left|\frac{d \bmod x}{dt}\right| \leqq \bmod \frac{dx}{dt} ;$$

c) $\bmod \int_\tau^t x dt \leqq \left|\int_\tau^t \bmod x dt\right|$.

To prove a) it is only necessary to apply the identity

$$\sum_{i=1}^r x_i^2 \cdot \sum_{i=1}^r y_i^2 = \left(\sum_{i=1}^r x_i y_i\right)^2 + \sum_{i,j=1}^r (x_i y_j - x_j y_i)^2, \tag{18}$$

after squaring both sides of the inequality. To prove b) two kinds of t-points must be considered, according as $x \neq 0$ or $x = 0$. For the former [see 5)]

$$\left|\frac{d \bmod x}{dt}\right| = \left|\frac{x \dfrac{dx}{dt}}{\bmod x}\right| \leqq \bmod \frac{dx}{dt},$$

on account of (18) when $y = \dfrac{dx}{dt}$; and for the latter the forward derivative by the mean value theorem is

$$\frac{d \bmod x}{dt} = \lim_{h=0} \frac{1}{h} \sqrt{\sum_{i=1}^r [h x_i'(t + \theta h)]^2} = + \sqrt{\sum_{i=1}^r \left(\frac{dx}{dt}\right)^2},$$

(see footnote p. 54) where h approaches zero from the positive side. From b) it follows further that

$$\left|\frac{d}{dt} \bmod \int_\tau^t x dt\right| \leqq \bmod x = \frac{d}{dt} \int_\tau^t \bmod x dt.$$

Therefore

$$\frac{d}{dt}\left\{\bmod \int_\tau^t x dt - \int_\tau^t \bmod x dt\right\} \leqq 0, \quad \frac{d}{dt}\left\{\bmod \int_\tau^t x dt + \int_\tau^t \bmod x dt\right\} \geqq 0,$$

and by using the former for $t > \tau$ and the latter for $t < \tau$, it is seen that c) must be true on account of the footnote on p. 55.

6. Systems of differential equations. By means of the definitions of §5 a system of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots, x_r), \qquad i = 1, 2, \cdots, r), \tag{19}$$

can be represented by a single equation of the form (1), where however x and f are complex with the elements x_i and f_i respectively. It is presupposed that the functions f_i are continuous in the interior of a region R of $r+1$ dimensions, and that a positive constant k' exists for any finite closed region R' interior to R, such that

$$\text{mod}[f(t, x) - f(t, y)] \leqq k' \text{mod}(x - y) \tag{20}$$

whenever (t, x), (t, y), and all the points (t, z) for which the elements z_i are in the interval $[x_i, y_i]$, lie in the region R'.*

If ξ is a complex number and (τ, ξ) is any point interior to R, a positive constant ρ can always be found such that all points (t, x) satisfying

$$R_1: \qquad |t - \tau| \leqq \rho, \quad \text{mod}(x - \xi) \leqq \rho,$$

are also interior points. The set of approximation functions for the system (19) is defined by equations of the same form as equations (3) but with $x^{(n-1)}$, $x^{(n)}$, ξ, and f complex. If t remains on the interval

$$|t - \tau| \leqq l, \dagger \tag{21}$$

where l is the smaller of ρ and $\dfrac{\rho}{M_1}$, M_1 being the maximum of mod f‡ in the

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*The analogue to the Lipschitz condition of §1 would be, in the notation of complex numbers [see §4, (8), (10), and (3)],

$$|f(t, x) - f(t, y)| \leqq K\, |x - y|, \tag{21}$$

where K is a matrix of positive constants k_{ij}. The condition given in the text is a consequence of this, as can be shown by taking the moduli of each side of the equation (21) and applying the relation

$$\left|\sum_{i,j=1}^{r} a_{ij} x_i x_j\right| \leqq \sum_{i,j=1}^{r} |a_{ij}| \sum_{i=1}^{r} x_i^2,$$

which holds for any quadratic form because $|x_i| \Big/ \sqrt{\sum_{j=1}^{r} x_j^2} \leqq 1$.

In case the elements f_i have continuous partial derivatives with respect to the variables x_j, the value of k_{ij} can be taken equal to the maximum of the absolute value of $f_{ij} = \dfrac{\partial f_i}{\partial x_j}$ $(i, j = 1, 2, \ldots, r)$ in the region R. For

$$f_i(t, x_1, \ldots, x_r) - f_i(t, y_1, \ldots, y_r) = \sum_{j=1}^{r} (x_j - y_j) f_{ij}\big(t, y_1 + \theta_1(x_1 - y_1), \ldots, x_r + \theta_r(x_r - y_r)\big),$$

where the θ's are all between zero and unity.

†It should be noticed that if the system of equations is linear in the x_i's this restriction on t is unnecessary. The approximation functions are well-defined on any interval for which the coefficients of the x_i's are continuous functions of t. (See §6.)

‡In many cases the proofs for a system of equations can be worded as for a single equation provided that absolute values are replaced by moduli as defined in §4.

region R_1, then by means of §5 c) it follows step by step that the functions $x^{(n)}$ satisfy the relations

$$\operatorname{mod}(x^{(n)} - \xi) \leqq M_1 |t - \tau| \leqq \rho,$$

and are well-defined and continuous in the interval (21).

In a manner analogous to that of §1, it can be shown with the help of (20) and §5 c) that the terms of the series

$$(22) \qquad \operatorname{mod}(x^{(1)} - \xi) + \operatorname{mod}(x^{(2)} - x^{(1)}) + \operatorname{mod}(x^{(3)} - x^{(2)}) + \cdots$$

are less than the corresponding terms of

$$\frac{N}{k_1}\left\{k_1|t-\tau| + \frac{k_1^2|t-\tau|^2}{2\,!} + \frac{k_1^3|t-\tau|^3}{3\,!} + \cdots\right\},$$

where N is the maximum of mod $f(t, \xi)$ in the interval (21), and the series (22) converges therefore uniformly on the interval (21). The r series defined by

$$(23) \qquad x = \xi + (x^{(1)} - \xi) + (x^{(2)} - x^{(1)}) + \cdots$$

are all convergent, and as in §1,

$$(24) \qquad x = \xi + \int_\tau^t f(t, x)\,dt.$$

Hence *the series* (23) *define r functions $x_i(t)$ which are continuous on the interval $|t-\tau| \leqq l$, have the values ξ_i for $t = \tau$, and satisfy the differential equations* (19).

These solutions can be continued to the right and left of the interval $[\tau - l, \tau + l]$, and the values of t which can be reached by such extensions have a lower bound t_0 and an upper bound t_1 as before, provided the region R is finite and closed.

Through any point $(\tau, \xi_1, \xi_2, \cdots, \xi_r)$ interior to the region R in which the properties of the functions f_i are defined, there passes a system of solutions of the equations (19), *denoted by*

$$(25) \qquad x_i = \phi_i(t, \tau, \xi_1, \xi_2, \cdots, \xi_r) \qquad (i = 1, 2, \cdots, r),$$

or, in the notation of complex numbers,

$$x = \phi(t, \tau, \xi).$$

The functions ϕ_i are continuous in t and define points $(t, x_1, x_2, \cdots, x_n)$ in the interior of R for all values of t in an interval

$$t_0 < t < t_1$$

including the value τ.

The statements made in §1 *concerning the behavior of the solution in the vicinity of* t_0 *and* t_1 *apply also to the solutions* (25).

7. Uniqueness and continuity for a system of equations. The region R_2 (compare §2) in the neighborhood of a system S of solutions $x = x(t)$ which are defined and are interior to R in an interval $[t', t'']$, consists of all the points (t, x) satisfying the inequalities

$$t' \leqq t \leqq t'', \quad \operatorname{mod}[x - x(t)] \leqq \rho.$$

For a properly chosen ρ, R_2 will be entirely interior to R. If $\bar{S}\Big(y = y(t)\Big)$ is another system of solutions interior to R_2 for $t = \tau$, then by §5 b) and (20),

$$\left|\frac{d}{dt}\operatorname{mod}(x - y)\right| \leqq \operatorname{mod}\frac{d(x-y)}{dt} = \operatorname{mod}[f(t, x) - f(t, y)]$$
$$\leqq k_2 \operatorname{mod}(x - y),$$

where k_2 belongs to R_2. As in §2, it follows that

$$\operatorname{mod}(x - y) \leqq e^{k_2|t-\tau|} \cdot \operatorname{mod}(\xi - \eta), \tag{26}$$

where ξ and η are complex numbers representing the values of x and y at $t = \tau$. But mod $(x - y)$ is necessarily $\geqq 0$. Hence *only one system of solutions of equations* (19) *can pass through a given point* $(\tau, \xi_1, \xi_2, \cdots, \xi_n)$ *interior to* R. *The functions* (25) *represent therefore all systems of solutions in the region.*

Let the system $x = x(t)$ be replaced by the system of solutions (25), and let R_2 be so chosen that the interval $[t', t'']$ includes the value τ. If $\Delta\tau$ and a complex number $\Delta\xi = (\Delta\xi_1, \Delta\xi_2, \cdots, \Delta\xi_n)$ are such that the point $(\tau + \Delta t, \xi + \Delta\xi)$ is interior to R_2, then from (26) when τ is replaced by $\tau + \Delta\tau$,

$$\operatorname{mod}[\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)]$$
$$\leqq e^{k_2|t-\tau-\Delta\tau|}\operatorname{mod}[\xi + \Delta\xi - \phi(\tau + \Delta\tau, \tau, \xi)];$$

or by means of (24),

$$\operatorname{mod}[\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)]$$
$$\leqq e^{k_2|t-\tau-\Delta\tau|}\operatorname{mod}\left[\Delta\xi - \int_\tau^{\tau+\Delta\tau} f(t, \phi)\,dt\right].$$

But by §4 a) and c), this becomes

$$\operatorname{mod}[\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)] \tag{27}$$
$$\leqq e^{k_2|t-\tau-\Delta\tau|}[\operatorname{mod}\Delta\xi + M_2|\Delta\tau|],$$

where M_2 is the maximum of mod f in R_2. The inequality (27) holds through-

out the interval T including τ, in which the solution $\bar{S}$,

$$x = \phi(t, \tau + \Delta\tau, \xi + \Delta\xi) ,$$

through the point $(\tau + \Delta\tau, \xi + \Delta\xi)$ is defined and interior to R_2.

Choose now $\Delta\tau$ and $\Delta\xi$ so that

$$(28) \qquad e^{k_2|t''-t'|}[\operatorname{mod}\Delta\xi + M_2|\Delta\tau|] < \rho.$$

The solution $\bar{S}$ can then be continued over the whole interval $[t', t'']$ because, on account of (28), $\bar{S}$ remains interior to R_2 for all values of t between t' and t''. Furthermore if t and $t + \Delta t$ are both in $[t', t'']$,

$$\begin{aligned}\operatorname{mod}\Delta\phi &= \operatorname{mod}[\phi(t + \Delta t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)] \\ &\leqq M_2|\Delta t| + e^{k_2|t-\tau-\Delta\tau|}[\operatorname{mod}\Delta\xi + M_2|\Delta\tau|],\end{aligned}$$

by means of §5 a), (24), and (27), in a manner similar to that of §3. It is readily seen that *the solutions* (25) *are continuous for all values of t and the initial values* $(\tau, \xi_1, \xi_2, \cdots, \xi_r)$ *which define points of the solution interior to the region R.*

8. Systems of linear equations. The system of linear equations

$$(29) \qquad \frac{dz_i}{dt} = \sum_{j=1}^{r} a_{ij}z_j \qquad (i = 1, 2, \cdots, r),$$

can be represented in the notation of complex numbers by the single equation

$$\frac{dz}{dt} = Az$$

[see §5, 7), 9), 10)], where z is complex with elements z_i, and A is the matrix of elements a_{ij} $(i, j = 1, 2, \cdots, r)$. The coefficients a_{ij} are supposed to be continuous functions of t on an interval $t_0 \leqq t \leqq t_1$. By the methods of §§6, 7* *a matrix E can be determined, of which the columns are r systems of solutions of the equations* (29), *taking respectively the columns of the matrix*

$$(30) \qquad \begin{matrix} 1, & 0, & 0, & \cdots, & 0 \\ 0, & 1, & 0, & \cdots, & 0 \\ 0, & 0, & 1, & \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & 0, & \cdots, & 1 \end{matrix}$$

*The region R consists of all points (t, z) for which $t_0 \leqq t \leqq t_1$ and z is arbitrary. The constant k can be taken as r times the maximum of the absolute values of the a^{ij} in the interval $[t_0, t_1]$.

as initial values for $t = \tau$. The elements of E are continuous functions of t and τ when these values lie in the interval $[t_0, t_1]$.

From §5, 12), and the fact that the columns of E are systems of solutions of equations (29), it follows that

$$\frac{dE}{dt} = AE. \tag{31}$$

Consider the set of functions

$$z = E\zeta,$$

where ζ is a complex constant. By differentiation and application of (31) [see §5, 7), 13), 10), 12)],

$$\frac{dz}{dt} = \frac{dE}{dt}\zeta = AE\zeta = Az.$$

Hence *the system of integrals of equations (29) taking any given initial values ζ for $t = \tau$, and which by §7 is unique, can be expressed as the product of the matrix E and the initial values ζ,*

$$z = E\zeta.$$

In the next section it will be important to consider the solutions of linear equations in which the coefficients a_{ij} involve a number of parameters besides t. Use will be made of the following

AUXILIARY THEOREM: *If the coefficients a_{ij} of the equations (29) involve also s parameters $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s)$, and are continuous functions of them in a finite closed region P, then the elements of the matrix E are continuous functions of t, τ, α for all values of t, τ, α such that t and τ are in the interval $[t_0, t_1]$ and α in P.*

The proof is easily made by examining the set of approximation functions of the form (3). Each of them is defined on the whole interval $[t_0, t_1]$ (see footnote p. 60), and is a continuous function of t, τ, α. If the constant N of §6 is taken as the maximum of mod $f(t, \zeta) =$ mod $A\zeta$ for all values of t in $[t_0, t_1]$ and of α in P, then the convergence proof is independent of the parameters α, and the series corresponding to (23) are uniformly convergent series of continuous functions of t, τ, α: which proves the theorem.*

9. Partial derivatives of a system of solutions. The existence of the first partial derivatives of the solutions (25) with respect to the initial

*See Jordan, *loc. cit.* p. 311.

constants can be proved, provided that the functions $f_i(t, x_1, x_2, \cdots, x_n)$ have continuous first partial derivatives with respect to the x's in the interior of the region R. Let $\bar{t}$, $\bar{\tau}$, $\bar{\xi}$ define a point of (25) interior to R, and suppose R_2 constructed as in §7 so that the interval $[t', t'']$ includes the values $\bar{t}$ and $\bar{\tau}$. The difference of the two solutions ϕ, $\phi + \Delta\phi$ corresponding to $(\bar{\tau}, \bar{\xi})$ and $(\bar{\tau} + \Delta\tau, \bar{\xi} + \Delta\xi)$ respectively, satisfies the equation

$$\frac{d\Delta\phi}{dt} = f(t, \phi + \Delta\phi) - f(t, \phi) = A\Delta\phi, \tag{32}$$

where the elements of the matrix A are determined by the equations

$$\begin{aligned}\Delta\phi_j \cdot a_{ij} = f_i(t, \phi_1, \cdots, \phi_{j-1}, \phi_j + \Delta\phi_j, \cdots, \phi_r + \Delta\phi_r) \\ - f_i(t, \phi_1, \cdots, \phi_{j-1}, \phi_j, \phi_{j+1} + \Delta\phi_{j+1}, \cdots, \phi_r + \Delta\phi_r).\end{aligned}$$

If $\bar{\tau}$ and $\bar{\xi}$ are considered fixed for the moment, it can be shown (compare with §4) that the coefficients a_{ij} are continuous functions of t, $\Delta\tau$, $\Delta\xi$, provided that

$$t' \leqq t \leqq t'', \quad |\Delta\tau| = |\tau - \bar{\tau}| \leqq \delta, \quad \operatorname{mod}\Delta\xi = \operatorname{mod}(\xi - \bar{\xi}) \leqq \delta, \tag{33}$$

where δ is so chosen that the point $(\bar{\tau} + \Delta\tau, \bar{\xi} + \Delta\xi)$ is in R_2 and the inequality (28) is satisfied. This follows from the fact that the solutions $\phi + \Delta\phi$ corresponding to such values of $\Delta\tau$, $\Delta\xi$ are defined and interior to R_2 in the interval $[t', t'']$, and according to §7 the functions ϕ are continuous.

By the auxiliary theorem of §8 a matrix E exists for the equations (32), with elements which take the initial values (30) at $t = \bar{\tau}$, and which are continuous functions of t, $\Delta\tau$, $\Delta\xi$ in the region (33). Suppose that $\Delta\xi_1$ is different from zero, while $\Delta\tau = \Delta\xi_2 = \Delta\xi_3 = \cdots = \Delta\xi_r = 0$. The r quotients $\dfrac{\Delta\phi}{\Delta\xi_1}$ [see §5, 6)] are a system of integrals of equations (32) with the initial values $(1, 0, 0, \cdots, 0)$ for $t = \bar{\tau}$, and therefore coincide with the first column of E. A similar reasoning applies to the other systems $\dfrac{\Delta\phi}{\Delta\xi_j}$ $(j = 2, \cdots, r)$ for which $\Delta\tau$ and all the elements of $\Delta\xi$ except $\Delta\xi_j$ are set equal to zero. Since the elements of E are continuous functions, it follows for each system that the limits of the quotients $\dfrac{\Delta\phi_i}{\Delta\xi_j}$ as $\Delta\xi_j$ approaches zero exist, and at the limit

$$\frac{\partial \phi}{\partial \xi} = E\Big|^{\Delta\tau = \Delta\xi = 0}, \tag{34}$$

where $\frac{\partial \phi}{\partial \xi}$ is the matrix of derivatives $\frac{\partial \phi_i}{\partial \xi_j}$ $(i, j = 1, 2, \cdots, r)$.

So far only the existence of the derivatives (34) has been proved, but it can be readily shown that they are also continuous at the given point $\bar{t}, \bar{\tau}, \bar{\xi}$. By substituting $x = \phi$ in the equations (19) and deriving successively for ξ_i $(i = 1, 2, \cdots, r)$, it follows that the columns of the matrix $\frac{\partial \phi}{\partial \xi}$ are solutions of the equations

$$\frac{dz}{dt} = f_x z, \tag{35}$$

where f_x denotes the matrix of elements $\frac{\partial f_i}{\partial x_j}$ $(i, j = 1, 2, \cdots, r)$ in which the arguments are $t, \phi(t, \bar{\tau}, \bar{\xi})$. Equations (32) go over into these equations when $\Delta\tau = \Delta\xi = 0$. If the $\bar{\tau}$ and $\bar{\xi}$ are replaced by $\tau = \bar{\tau} + \Delta\tau$ and $\xi = \bar{\xi} + \Delta\xi$, the coefficients $\frac{\partial f_i}{\partial x_j}$ will be continuous functions of t, τ, ξ in the region (33). The elements of the matrix of solutions $\frac{\partial \phi}{\partial \xi}$ with the initial values (30) at $t = \tau$, are therefore continuous functions of t, τ, ξ in the region (33), in particular at the point $(\bar{t}, \bar{\tau}, \bar{\xi})$.

When $\Delta\tau \neq 0$, $\Delta\xi = 0$, the value of $\Delta\phi$ at $t = \bar{\tau}$ is found from equation (24) to be

$$\Delta\phi\Big|^{t=\bar{\tau}} = \int_{\bar{\tau}+\Delta\tau}^{\bar{\tau}} f(t, \phi + \Delta\phi)dt = -\Delta\tau \cdot \bar{f}, \tag{36}$$

where the elements of the complex number $\bar{f}$ are

$$f_i\left(\bar{\tau} + \theta_i\Delta\tau,\ \phi(\bar{\tau} + \theta_i\Delta\tau,\ \bar{\tau} + \Delta\tau,\ \xi)\right), \quad 0 < \theta_i < 1, \ (i = 1, 2, \cdots, r).$$

Accordingly the quotients $\frac{\Delta\phi}{\Delta\tau}$, which form a system of solutions of equations (32) with initial values at $t = \tau$ defined by (36), have the values (see §8)

$$\frac{\Delta\phi}{\Delta\tau} = -E\bar{f}.$$

At the limit for $\Delta\tau = 0$,

$$\frac{\partial \phi}{\partial \tau} = -\frac{\partial \phi}{\partial \xi} f(\tau, \xi), \tag{37}$$

where the right member is the product of the matrix $\frac{\partial\phi}{\partial\xi}$ and the complex number $f(\tau, \xi)$ [see §5, 6) and 10)].

It has been proved therefore that *when the functions $f_i(t, x_1, x_2, \cdots, x_n)$ have continuous first partial derivatives with respect to the x_i's in the interior of the region R, then the solutions (25) have first partial derivatives which are continuous for all values $\bar{t}$, $\bar{\tau}$, $\bar{\xi}$ defining points of the solution interior to R.*

The existence of higher derivatives at a point $\bar{t}$, $\bar{\tau}$, $\bar{\xi}$ when further assumptions are made upon the functions f_i, can be proved by means of the $2r + 1$ equations,

$$\frac{d\tau}{dt} = 0, \quad \frac{d\xi}{dt} = 0, \quad \frac{dz}{dt} = f_x z, \tag{38}$$

where the arguments of f_x are t, $\phi(t\,\tau, \xi)$. The right members are functions of t, τ, ξ, z, and are continuous when t, τ, ξ satisfy the conditions (33) whatever the value of the arguments z. If the f_i have continuous second partial derivatives with respect to the x_i's, then according to the last theorem the right members of (38) have continuous first partial derivatives for τ, ξ, z, and their solutions

$$\tau = \tau, \quad \xi = \xi, \quad z = \frac{\partial\phi}{\partial\xi_i}$$

are therefore differentiable for the initial values τ, ξ. If furthermore the f_i have continuous first partial derivatives with respect to t, then on account of the differentiability of the functions $\frac{\partial\phi_i}{\partial\xi_j}$ and f_i, the expressions (37) can be differentiated once, and ϕ twice, for τ. From (24) the functions φ can also be differentiated twice for t. In general the following theorem can be proved:

If the functions $f_i(t, x_1, x_2, \cdots, x_r)$ have all the partial derivatives of order $n - 1$, and all of order n except perhaps the n^{th} derivatives with respect to t alone, continuous in the region R, then the solutions (25) of equations (19) have continuous partial derivatives of the n^{th} order for all values of t, τ, ξ defining points of the solution (25) interior to R.

For suppose that it holds for n, and that the functions f_i have all derivatives of the n^{th} order and all of order $n + 1$ except perhaps the $(n + 1)^{\text{st}}$ derivatives with respect to t. Then according to the hypothesis the functions ϕ have continuous n^{th} derivatives, and, on account of the existence of the $(n + 1)^{\text{st}}$ derivatives of the functions f_i, the right members of equations (38)

can be differentiated n times for t, τ, ξ, and z. The solutions $\frac{\partial \phi}{\partial \xi}$ and the expressions (37) can therefore also be differentiated n times. Since the functions f_i have continuous n^{th} derivatives with respect to t alone, equation (24) shows that the functions ϕ_i can be differentiated $n+1$ times for t alone. The induction is therefore complete, and the theorem holds for any value of n.

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ON THE CONFORMAL REPRESENTATION OF CERTAIN ISOSCELES TRIANGLES UPON THE UPPER HALF PLANE

BY L. WAYLAND DOWLING

1. Introduction. The problem of conformally representing a polygon whose sides are right lines or arcs of circles, upon the upper half of the z-plane, was completely solved more than thirty years ago by Christoffel and Schwarz.*

If the angles of a rectilinear triangle, situated in the w-plane, are $\alpha\pi$, $\beta\pi$, and $\gamma\pi$, then the representation is effected by the formula

$$C_1 w + C_2 = \int \frac{dz}{(z-a)^{1-\alpha}(z-b)^{1-\beta}(z-c)^{1-\gamma}},$$

where the constants a, b, and c, are the values of z (real) corresponding to the vertices of the triangle. The constants C_1 and C_2 depend upon the position of the triangle in the w-plane.

In the following pages this formula is applied to certain plane triangles, in particular, isosceles triangles, and the discussion is limited to the simplest cases, viz: triangles whose angles are rational proper fractions of 180°. Obviously any such triangle can be formed from a regular polygon by joining three of its vertices, provided the number of sides of such a polygon is a common multiple of the denominators of α, β, and γ.

It is assumed, for convenience, that the constants a, b, and c, have the values -1, $+1$, and 0 respectively, and that the lowest common denominator of the fractions α, β, and γ, is n, so that the polygon of *least* number of sides from which the triangle can be formed is an n-gon. Furthermore, putting $1-\alpha=\lambda_1/n$, $1-\beta=\lambda_2/n$, $1-\gamma=\lambda_3/n$, the Christoffel-Schwarz formula becomes

$$C_1 w + C_2 = \int \frac{dz}{\sqrt[n]{(z+1)^{\lambda_1}(z-1)^{\lambda_2}z^{\lambda_3}}}. \tag{1}$$

In the formula thus written, λ_1, λ_2, and λ_3 are integers satisfying the conditions

$$\Sigma\lambda_i = 2n;\ 1 < \lambda_i < n;\ (i = 1, 2, 3). \tag{2}$$

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* Christoffel: *Ann. di. mat.*, ser. 2, vol. 1 (1867), p. 97.
Schwarz: *Crelle*, vol. 70 (1869), p. 117; *Werke*, vol. 2, p. 80.

Every set of three integers satisfying these conditions may be taken to represent a triangle; the symbol $[\lambda_1, \lambda_2, \lambda_3]$ designating the triangle whose angles are

$$\left(1-\frac{\lambda_1}{n}\right)\pi, \quad \left(1-\frac{\lambda_2}{n}\right)\pi, \quad \text{and} \quad \left(1-\frac{\lambda_3}{n}\right)\pi.$$

The representation is obviously not unique, for any given triangle will be represented by an infinity of such sets of numbers just as it can be formed from an infinity of regular polygons by joining vertices. Any given set of numbers, however, satisfying conditions (2), can represent but one triangle.

The integral in (1) is an Abelian integral of the first kind existing upon the n-leaved Riemann surface

$$(3) \qquad u^n = (z+1)^{\lambda_1}(z-1)^{\lambda_2}z^{\lambda_3}.$$

This surface has winding points at $z = -1, +1$, and 0, and nowhere else. If $p_i = [n; \lambda_i]$, *i. e.*, the highest common factor of n and λ_i, then, at the winding point for which λ_i is the exponent, the n leaves are grouped into p_i groups, each group consisting of n/p_i leaves winding together. Hence the deficiency of the surface is

$$(4) \qquad p = \frac{\Sigma p_i\left(\frac{n}{p_i}-1\right)}{2} - n + 1 = \frac{n+2-\Sigma p_i}{2}.$$

Besides the integral (1) there are $p-1$ other integrals of the first kind existing upon the same surface and linearly independent of (1).

2. The complete set of integrals of the first kind. The first question will be to find out the meaning which such a set of independent integrals has in the problem of the conformal representation of the given triangle $[\lambda_1, \lambda_2, \lambda_3]$. The theory of Abelian integrals teaches that, for proper values of the integers k, s_1, s_2, s_3, the integral

$$(5) \qquad I_k = \int \frac{(z+1)^{s_1}(z-1)^{s_2}z^{s_3}}{u^k}\, dz \equiv \int \frac{dz}{\sqrt{(z+1)^{\mu_1}(z-1)^{\mu_2}z^{\mu_3}}}$$

will be an integral of the first kind upon the same surface (3) and will be independent of (1). The necessary and sufficient conditions to be imposed upon the μ's are:

$$(6) \qquad k\lambda_i \equiv \mu_i \pmod{n}, \ \Sigma\mu_i = 2n, \ 1 < \mu_i < n; \ (i = 1, 2, 3).$$

The first of these conditions states that the μ's are derived from the *given* λ's as is indicated in (5). The conditions $\Sigma\mu_i = 2n$ and $1 < \mu_i < n$ insure the finiteness of the integral over the entire Riemann surface.

It is necessary to show, first of all, that the integrals I_k form a *complete set* of linearly independent integrals when the conditions (6) are satisfied. That the integrals I_k are linearly independent follows from the theory as above noted.

There are obviously not more than $n - 1$ different sets of μ's satisfying the conditions $k\lambda_i \equiv \mu_i \pmod{n}$ and $1 < \mu_i < n$. In fact if n is prime to λ_1, λ_2, and λ_3 there will be *just* $n - 1$ sets of μ's satisfying the above congruences and the conditions $1 \leqq \mu_i < n$. But if n is not prime to each of the λ's and if, as before, p_i stands for $[n; \lambda_i]$, then the theory of linear congruences shows that there are $p_i - 1$ values of k other than zero for which $k\lambda_i \equiv 0$ (mod n); *i. e.*, there will be $p_i - 1$ of the μ_i's equal to zero and these must be thrown out as not satisfying the conditions $1 < \mu_i < n$. Hence there cannot be more than $n - 1 - \Sigma(p_i - 1)$ sets of μ's satisfying the first and third of conditions (6). Of these sets, however, not all will satisfy the remaining condition, $\Sigma\mu_i = 2n$. For, suppose k_1 is a multiplier giving rise to a set of μ's satisfying all the conditions (6), then the multiplier $n - k_1$ will give rise to a set of μ's not satisfying the condition $\Sigma\mu_i = 2n$, since $(n - k_1)\lambda_i$ is obviously congruent to $n - \mu_i \pmod{n}$ if $k_1\lambda_i \equiv \mu_i \pmod{n}$. But $\Sigma(n - \mu_i) = n$. Hence there cannot be more than

$$\frac{(n-1) - \Sigma(p_i - 1)}{2} = \frac{n + 2 - \Sigma p_i}{2} = p$$

sets of μ's satisfying all of the conditions (6). To see that there is exactly this number notice that for every set of μ's satisfying the congruences $k\lambda_i \equiv \mu_i \pmod{n}$ we must have $\Sigma\mu_i \equiv 0 \pmod{n}$ because $\Sigma\lambda_i = 2n$. Imposing the conditions $1 < \mu_i < n$ we see that $\Sigma\mu_i$ must be either n or $2n$, and for every value of k that makes $\Sigma\mu_i = 2n$ there is another, namely $n - k$, that makes $\Sigma\mu_i = n$, and *vice versa*. Therefore there are exactly p sets of integers μ satisfying conditions (6) and hence exactly p of the integrals I_k; *i. e.*, the I_k's form a *complete* set of linearly independent integrals upon the Riemann surface (3).

As an example, consider the triangle [7, 8, 11] whose angles are $\frac{6}{13}\pi$, $\frac{5}{13}\pi$, $\frac{2}{13}\pi$. It is represented conformally upon the upper half plane

by the integral

$$I_1 = \int \frac{dz}{\sqrt[13]{(z+1)^7 (z-1)^8 z^{11}}}.$$

The integrals I_1, I_3, I_7, I_8, I_9, and I_{11} form a complete set of linearly independent integrals upon the surface $u^{13} = (z+1)^7 (z-1)^8 z^{11}$, any one of which may be written down at once, for instance:

$$I_8 = \int \frac{(z+1)^4 (z-1)^4 z^6\, dz}{u^8}.$$

Now the μ's corresponding to any one of the I_k's must satisfy precisely the conditions (2) imposed upon the λ's and hence it follows that each integral I_k represents some triangle conformally upon the upper half-plane; moreover, this triangle is formed from exactly the same regular polygon as was the original triangle. Therefore:

The complete set of linearly independent integrals of the first kind, I_k, existing upon the Riemann surface $u^n = (z+1)^{\lambda_1} (z-1)^{\lambda_2} z^{\lambda_3}$ represents conformally upon the upper half-plane a set of triangles each of which can be formed from the same regular n-gon by joining three of its vertices in the proper manner.

This theorem sets up an intimate relation between a particular set of linearly independent integrals of the first kind existing upon the Riemann surface (3) and a particular set of triangles formed from a regular n-gon. It does not assert, however, that the triangles will necessarily be all distinct. In fact, the same triangle will, in general, be represented conformally upon the upper half-plane by different integrals if its vertices be permuted. In the examples quoted above we have the correspondences

$$\begin{array}{llll} I_1 \cdots [\ 7,\ 8,\ 11], & \qquad & I_7 \cdots [10,\ 4,\ 12], \\ I_3 \cdots [\ 8,\ 11,\ 7], & & I_8 \cdots [\ 4,\ 12,\ 10], \\ I_9 \cdots [11,\ 7,\ 8], & & I_{11} \cdots [12,\ 10,\ 4]. \end{array}$$

Here, I_1, I_3, and I_9 represent the same triangle with its vertices permuted cyclically, and the same is true of the integrals I_7, I_8, I_{11}.

3. The regular n-gon. A second question would be to discuss completely the triangles and corresponding integrals arising from any given n-gon. The general problem, however, is one of great complexity and does not seem to admit of concise statement or solution. Something may be said, however, and, first of all, the number of distinct triangles that can be formed from the

n-gon by joining vertices may be determined. For this purpose it is convenient to arrange the triangles in columns as follows:

$$
\begin{array}{llll}
[n-1, n-1, 2] & [n-2, n-2, 4] & \cdots & [n-k, n-k, 2k] \\
[n-1, n-2, 3] & [n-2, n-3, 5] & \cdots & [n-k, n-k-1, 2k+1] \\
\quad\cdots & \quad\cdots & & \quad\cdots \\
[n-1, n-r, r+1] & [n-2, n-s, s+2] & \cdots & [n-k, n-L, k+L] \\
\quad\cdots & \quad\cdots & & \quad\cdots
\end{array}
$$

The columns are to be continued until the λ_2, λ_3 of the last triangle in each are either equal or $\lambda_2 = \lambda_3 + 1$. In this way the scheme takes account of every possible distinct triangle without repetitions. Suppose the columns numbered from left to right in order and divided into two groups; group I containing the odd numbers and group II, the even numbers. The following statements are then easily verified:

1. The number of columns is always the greatest integer contained in $n/3$. For the number of columns is evidently equal to k, and $2k$ must be $\leqq$ to $n - k$, hence $k \leqq n/3$.
2. If $n \equiv 0$, 1, or 2 (mod 6), the number of columns in each group will be equal; while if $n \equiv 3$, 4, or 5 (mod 6), the number of columns in group I will exceed those in group II by unity.
3. The number of triangles in the columns of either group form an arithmetical progression whose common difference is -3. If n is even, the first term of this progression for group I is $\frac{1}{2}(n-2)$; for group II, $\frac{1}{2}(n-4)$. If n is odd the first term for group I is $\frac{1}{2}(n-1)$; for group II, $\frac{1}{2}(n-5)$.

These statements furnish the means for computing the number of triangles by a simple summation of the series involved. If ν represents the total number of triangles, the results may be concisely stated as follows:—

$$\text{If } n^2 \equiv 0 \pmod 6, \qquad \nu = \frac{n^2}{12};$$

$$\text{if } n^2 \equiv 1 \pmod 6, \qquad \nu = \frac{n^2-1}{12};$$

$$\text{if } n^2 \equiv 3 \pmod 6, \qquad \nu = \frac{n^2+3}{12};$$

$$\text{if } n^2 \equiv 4 \pmod 6, \qquad \nu = \frac{n^2-4}{12}.$$

Among the ν possible distinct triangles that can be formed from the given n-gon, there is one group which stands out prominently from the rest. This group contains all the possible distinct isosceles triangles that can be formed from the n-gon, and is studied in detail in the following section.

4. The isosceles triangles. The integral (1) may be written in the form

$$C_1 w + C_2 = \int \frac{\frac{dz}{z^2}}{\sqrt[n]{\left[\frac{z+1}{z}\right]^{\lambda_1}\left[\frac{z-1}{z}\right]^{\lambda_2}}}, \tag{7}$$

since $\lambda_1 + \lambda_2 + \lambda_3 = 2n$. The triangle is assumed to be isosceles, so that $\lambda_1 = \lambda_2$. Further, making the substitution

$$\frac{z^2 - 1}{z^2} = t^n,$$

the integral reduces to

$$C_1 w + C_2 = \frac{n}{2}\int \frac{t^{n-1-\lambda_1}\,dt}{\sqrt{1 - t^n}}, \tag{8}$$

which is hyperelliptic in general.

Putting $\lambda_1 = n - r$ so that $r\pi/n$ is one of the two equal angles of the triangle, and allowing C_1 and C_2 to be $n/2$ and 0 respectively, the integral becomes

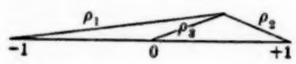
$$w = \int^t \frac{t^{r-1}\,dt}{\sqrt{1 - t^n}}.$$

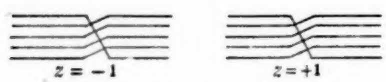
Hence the theorem: — *An isosceles triangle in the w-plane whose angles are $r\pi/n$, $r\pi/n$, $(n - 2r)\pi/n$ is represented conformally upon the upper half of the z-plane by the equations*

$$w = \int^t \frac{t^{r-1}\,dt}{\sqrt{1 - t^n}}, \quad t^n = \frac{z^2 - 1}{z^2}. \tag{9}$$

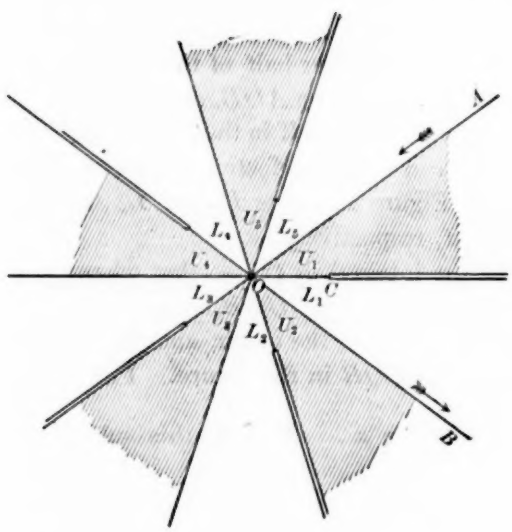
Proceeding to the geometric significance of these equations, there is, first of all, an n-leaved Riemann surface spread over the z-plane. This surface can be rendered simply-connected by a cut and proper bridging along the segment of the real axis extending from -1 through 0 to $+1$. This surface is transformed, by the substitution $\frac{z^2 - 1}{z^2} = t^n$, into a two-leaved Riemann surface

spread over the t-plane and having winding points at the places representing the n roots of unity. Infinity is a winding point or not according as n is odd

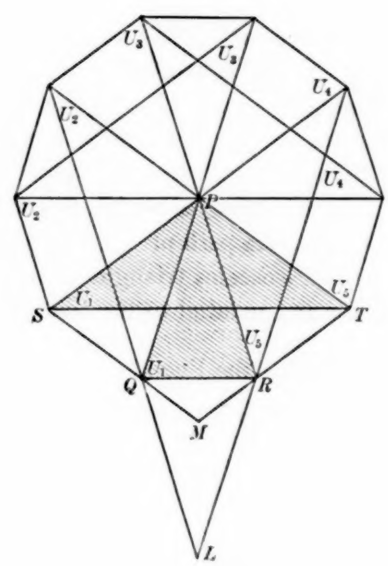




z-plane, a Riemann surface with five leaves. $p = 2$.



t-plane, a Riemann surface with two leaves. $p = 2$.



w-plane.

FIG. 1.

or even. This surface may be made simply connected by cuts extending radially from the finite winding points to infinity. Consider, next, what corresponds to the real z-axis.

Putting

$$z + 1 = \rho_1 e^{i\theta_1}, \quad z - 1 = \rho_2 e^{i\theta_2}, \text{ and } z = \rho_3 e^{i\theta_3},$$

then

$$t = \sqrt[n]{\frac{\rho_1\rho_2}{\rho_3^2}}\, e^{i\left(\frac{\theta_1 + \theta_2 - 2\theta_3}{n}\right)}.$$

If, now, z starts at 0 and describes the real axis in the positive direction, one of the values of t will start at ∞ and describe the line inclined to the axis of reals at an angle π/n. The segment of the z-axis between 0 and 1 corresponds to the line AO [see Figure 1]. The segment of the z-axis from 1 to ∞ corresponds to the line OC from $t = 0$ to $t = 1$. Here t encounters the cut extending from 1 to ∞ and descends to the lower leaf returning to 0 as z describes the segment from ∞ to -1. Finally, as z describes the segment from -1 to 0, t describes the line OB inclined to the real axis at an angle $-\pi/n$ and extending, on the lower leaf, from 0 to ∞. The upper half of the first leaf, say, of the z-surface is thus correlated to the infinite sector AOB, one-half of which lies in the upper leaf of the t-surface and the other half in the lower leaf. The complete correlation of the two surfaces can now be effected by a simple inspection of the figure, as is indicated in Figure 1, where U_i and L_i stand for the upper and lower halves of the i^{th}-leaf, respectively.

Choosing ∞ for the lower limit of the integral in (9) so that when $t = \infty$, $w = 0$, the path of w corresponding to the t-path $AOCOB$ is found to be the given isosceles triangle situated with its vertex at the origin and its base below and parallel to the real w-axis, viz., at PQR in the figure. For when t describes the line AO, it is always equal to $e^{\frac{\pi i}{n}}x$. Making this substitution for t in the integral (9), it becomes

$$w = e^{\frac{r\pi i}{n}} \int_\infty^x \frac{x^{r-1}dx}{\sqrt{1 + x^n}}.$$

The integral

$$\int_\infty^x \frac{x^{r-1}dx}{\sqrt{1 + x^n}}$$

is obviously real and finite for all real positive values of x. Hence w describes a line inclined to the real w-axis at an angle $r\pi/n$.

The integrals

(10) $$\int_0^\infty \frac{x^{r-1}dx}{\sqrt{1 + x^n}} \text{ and } \int_0^1 \frac{x^{r-1}dx}{\sqrt{1 - x^n}},$$

for which we shall use the symbols R_r and K_r respectively, are expressible as Γ-functions and are real positive constants.

They are related, as the theory of the Γ-function shows, by the equation

$$R_r \cos \frac{r\pi}{n} = K_r. \tag{11}$$

It is easy to complete the w-path corresponding to the t-path $AOCOB$. The following table exhibits the correspondence:

$$\begin{array}{lllll} z = 0, & 1, & \infty, & -1, & 0; \\ t = \infty, & 0, & 1, & 0, & \infty; \\ w = 0, & -e^{\frac{r\pi i}{n}} R_r, & -iR_r \sin \dfrac{r\pi}{n}, & e^{-\frac{r\pi i}{n}} R_r, & 0; \end{array}$$

w thus passes around the triangle in the positive direction. The vertices of the triangle are at 0, $-e^{\frac{r\pi i}{n}} R_r$, $e^{-\frac{r\pi i}{n}} R_r$ and its angles are $(n-2r)\pi/n$, $r\pi/n$, $r\pi/n$, respectively.

The figure shows four other triangles corresponding to the upper halves of the four other leaves of the z-surface.

The number of linearly independent integrals of the form

$$\int \frac{t^{r-1}dt}{\sqrt{1-t^n}}$$

is $\frac{1}{2}(n-2)$ if n is even, and $\frac{1}{2}(n-1)$ if n is odd. Again, the number of distinct isosceles triangles that can be formed from the regular n-gon is $\frac{1}{2}(n-2)$ for n even, and $\frac{1}{2}(n-1)$ for n odd. Hence there is a one to one correspondence between these triangles and the hyperelliptic integrals. This result may be embodied in the following theorem:

The totality of distinct isosceles triangles that can be formed from a regular n-gon is represented upon the upper half of the z-plane by the complete set of linearly independent hyperelliptic integrals of the form

$$\int \frac{t^{r-1}dt}{\sqrt{1-t^n}};$$

t being connected with z by the relation

$$t^n = \frac{z^2 - 1}{z^2} \cdot$$

The isosceles triangles are not the only ones that can be treated by means of hyperelliptic integrals. In fact, every right triangle whose angles are

$$\frac{\pi}{2}, \ \frac{r\pi}{n}, \ \frac{\left(\frac{n}{2} - r\right)\pi}{n}$$

can be conformally represented upon the upper half of the z-plane by means of the relations

$$w = \int \frac{t^{\frac{r-2}{2}}\, dt}{\sqrt{1 - t^{\frac{n}{2}}}}, \quad 2t^{\frac{n}{2}} = \frac{z - 1}{z}. \tag{12}$$

This follows upon reducing integral (7) by the substitution $2t^{\frac{n}{2}} = \dfrac{z-1}{z}$, properly choosing constants, and afterwards assuming

$$\frac{\lambda_1}{n} = \frac{1}{2}, \ \lambda_2 = n - r.$$

The fact is apparent geometrically, since every right triangle is exactly one-half the corresponding isosceles triangle.

5. Special cases. In this section it is proposed to study in more detail the triangles formed from n-gons in which n has the values, 3, 4, $\cdots$ 10, and, in particular, to determine all the triangles whose conformal representation can be effected by hyperelliptic integrals of deficiency 2. The cases $n = 3$, 4 are well known and lead to elliptic integrals. The representation is effected in these cases by the relations*

$$w = \int_{\infty}^{t} \frac{dt}{\sqrt{1 - t^3}}, \quad t^3 = \frac{z^2 - 1}{z^2}, \tag{13}$$

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* Love: Vortex Motion in Certain Triangles, *Amer. Jour.*, vol. 11 (1889). The integral expressions and the orientation of the triangles are completely studied in this article.

and

$$w = \int_\infty^t \frac{dt}{\sqrt{1-t^4}}, \quad t^4 = \frac{z^2-1}{z^2},$$

respectively.

In these cases the triangles completely cover the w-plane without overlapping.

For $n = 5$, we have $\nu = 2$. The two triangles are [4, 4, 2] and [3, 3, 4]; *i. e.*, they are both isosceles. The conformal representation is effected by the equations

$$(14) \qquad w_1 = \int_\infty^t \frac{dt}{\sqrt{1-t^5}}, \quad w_2 = \int_\infty^t \frac{tdt}{\sqrt{1-t^5}}, \text{ and } t^5 = \frac{z^2-1}{z^2}.$$

Figure 1 exhibits the correspondence, the triangles PST and PQR corresponding to the upper half of the first leaf of the z-surface. Here the w-plane cannot be covered completely without overlapping and it is natural to seek some surface that can be so covered.

For this purpose I choose two integrals:

$$(15) \qquad w_1^1 = \frac{1}{R_1}\int_\infty^t \frac{dt}{\sqrt{1-t^5}}, \quad w_2^1 = \frac{1}{R_2}\int_\infty^t \frac{tdt}{\sqrt{1-t^5}},$$

where R_1 and R_2 have the definitions given in (10). This causes the vertices S, T, and Q, R of the two triangles to lie on the unit circle; thus, if a is $e^{\frac{\pi i}{5}}$, then S and T are at a^6 and a^9, respectively, while Q and R lie at a^7 and a^8, respectively. If, now, v_1 and v_2 are two integrals connected with w_1^1 and w_2^1 by the equations

$$(16) \qquad \begin{aligned} a^6 v_1 + a^9 v_2 &= w_1^1, \\ a^7 v_1 + a^8 v_2 &= w_2^1 \end{aligned}$$

then the vertices of the two triangles will be given by the following values ot v_1 and v_2:

$$P: \begin{Bmatrix} v_1 = 0 \\ v_2 = 0 \end{Bmatrix}; \; S \text{ and } Q: \begin{Bmatrix} v_1 = 1 \\ v_2 = 0 \end{Bmatrix}; \; R \text{ and } T: \begin{Bmatrix} v_1 = 0 \\ v_2 = 1 \end{Bmatrix}.$$

Constructing the triangles SMT and QLR which correspond to the lower half of the first leaf of the z-surface, it is seen that the vertices M and L are given by the values $v_1 = 1$, $v_2 = 1$. This suggests the following geometric device: if the quadrilateral $PSMT$ be bent along the line PM and the quadrilateral $PQLR$ along the line QR, the two may be fitted together to form a tetrahedron $PQRL$ (Figure 2), whose vertices correspond, respectively, to the following values:

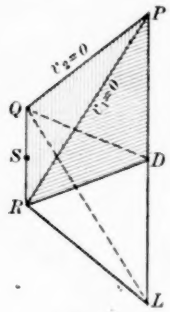

FIG. 2.

$$P:\begin{Bmatrix} v_1 = 0 \\ v_2 = 0 \end{Bmatrix}; \quad Q:\begin{Bmatrix} v_1 = 1 \\ v_2 = 0 \end{Bmatrix}; \quad R:\begin{Bmatrix} v_1 = 0 \\ v_2 = 1 \end{Bmatrix}; \quad L:\begin{Bmatrix} v_1 = 1 \\ v_2 = 1 \end{Bmatrix}.$$

The points S and D (where S bisects QR) both correspond to the values $v_1 = v_2 = \frac{1}{2}$.

To any *given* z in the upper half of the first leaf of the Riemann surface over the z-plane there correspond in general *two* points in the shaded portion of the tetrahedron, the exceptions being for $z = 0$, $+1$, or -1 which correspond respectively to P, Q, R. $z = \infty$ corresponds to the two points S and D. Five such tetrahedra, connected together at one vertex, form a complete picture of the z-Riemann surface.

For $n = 6$ we have $\nu = 3$. There are two isosceles triangles, [5, 5, 2] and [4, 4, 4], and one right triangle [5, 4, 3]. The right triangle is elliptic; *i. e.*, it is represented conformally upon the upper half of the z-plane by an elliptic integral. Taken in connection with the triangles noticed for $n = 5$ and 4, respectively, it completes the set of elliptic triangles.

The conformal representation of the isosceles triangles is effected by the equations

$$(17) \qquad w_1 = \int_{\infty}^{t} \frac{dt}{\sqrt{1 - t^6}}, \quad w_2 = \int_{\infty} \frac{tdt}{\sqrt{1 - t^6}}, \quad t^6 = \frac{z^2 - 1}{z^2}.$$

The tetrahedron formed as in the case for $n = 5$ is regular, showing clearly the possibility of representing the triangles conformally upon the upper half-plane by means of elliptic functions.* In fact, the substitution $t^2 = 1/s$

* Klein, *Vorlesungen über das Ikosaeder*.

in the first integral of (16), $t^2 = s$ in the second, with proper choice of constants, reduces each of them to the same elliptic integral; viz:

$$\int \frac{dy}{\sqrt{4y^3 - 4}}.$$

The correlation of the two triangles can then be effected completely by the relations

$$(18) \qquad p^3w = \frac{z^2}{z^2 - 1} \text{ and } p^3w = \frac{z^2 - 1}{z^2},$$

respectively.*

The places $z = 0, 1, -1$ which correspond to the vertices of the triangles are seen to be the zeros and infinities of the elliptic function pu, defined by the equation

$$p'u = \sqrt{4p^3u - 4}.$$

The period-parallelogram is a rhombus (Figure 3). If this rhombus be doubly-covered, cut and bridged along the line AB, and the two leaves sewed together along the outer edges, it may be completely covered without overlapping by alternately shaded and unshaded triangles of either sort, [5, 5, 2] or [4, 4, 4]. The rhombus, so constructed, will then be a complete picture of the corresponding z-Riemann surface, there being just twelve equal regions by either mode of division. That such a representation is possible, is at once apparent from the fact that both isosceles triangles are the doubles of the same elliptic triangle [5, 4, 3], and thus the square-root process in the complex-plane is suggested.†

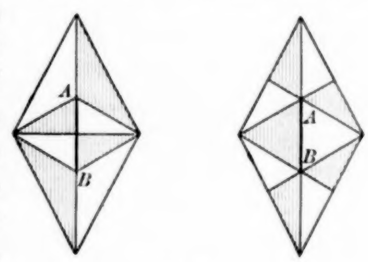

FIG. 3.

For $n = 7$, $\nu = 4$. There are three isosceles triangles which are conformally represented upon the upper half-plane by the three hyperelliptic integrals

$$\int_\infty^t \frac{t^{r-1}dt}{\sqrt{1 - t^7}} \ (r = 1, 2, 3) \text{ and } t^7 = \frac{z^2 - 1}{z^2}.$$

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* Love, *loc. cit.*

† This representation formed the basis of a report read before the Chicago section of the American Mathematical Society, January 2nd and 3rd, 1902. See *Bulletin* for February of that year

Besides these isosceles triangles, there is a "cyclic" triangle [6, 5, 3] represented conformally upon the upper half-plane by an Abelian integral of deficiency $p = 3$. The complete set of linearly independent integrals I_k [see §2] represents this same triangle with its vertices permuted. There is no triangle formed from the heptagon which can be conformally represented upon the upper half-plane by an integral of deficiency 2.

For $n = 8$, $\nu = 5$. There are three isosceles triangles one of which is elliptic. The other two lead to integrals of deficiency 3. There is a scalene triangle [7, 6, 3], which also leads to an integral of deficiency 3. There remains the right triangle [4, 5, 7] which can be conformally represented by an integral of deficiency 2: viz., from (1),

$$C_1 w + C_2 = \int \frac{dz}{\sqrt[8]{(z+1)^4(z-1)^5 z^7}} .$$

The second integral of the first kind existing upon the Riemann surface is

$$C_1 w + C_2 = \int \frac{dz}{\sqrt[8]{(z+1)^4(z-1)^7 z^5}} ,$$

which represents the same triangle with the last two vertices permuted, *i. e.*, the triangle [4, 7, 5]. These two integrals reduce to the normal hyperelliptic form by means of the substitution $2t^4 = \dfrac{z-1}{z}$, so that, by proper choice of the constants, the conformal representation is effected by the equations

$$(19) \qquad w_1 = \int_\infty^t \frac{dt}{\sqrt{t-t^5}}, \quad w_2 = \int_\infty^t \frac{t\,dt}{\sqrt{t-t^5}}, \quad 2t^4 = \frac{z-1}{z} .$$

These integrals also follow from formula (12).

Here we have an eight-leaved Riemann surface over the z-plane. The corresponding t-surface is two-leaved with winding points at 0, ∞, ± 1, and $\pm i$. The real axis of the first leaf, say, of the z-surface corresponds to the boundary of a sector in the first quadrant of the upper leaf of the t-surface whose vertical angle is $\pi/4$. The complete correspondence between the two surfaces is easily determined, but is not necessary for our present purpose. The w-triangles corresponding to the upper half of the first leaf of the z-surface have

the positions ABC and $AB'C'$, as shown in Figure 4. By completing the parallelograms, as shown in the figure, where the unshaded triangles correspond to the lower half of the upper leaf of the z-surface,* and choosing con-

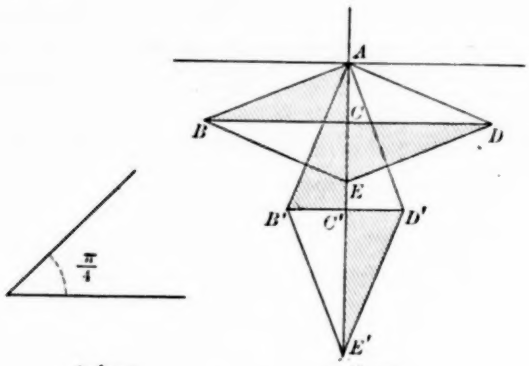

FIGURE 4.

stants properly as in the case for $n = 5$, the two parallelograms may be bent to form a tetrahedron which can be completely covered without overlapping by the triangles.

The tetrahedron can be covered by repetitions of either of the isosceles triangles, [7, 7, 2] or [5, 5, 6]. As a matter of fact each of these isosceles triangles is double the right triangle we have been discussing. The square-root process in the complex plane is again suggested. The conformal representation of the set of three isosceles triangles formed from the octagon is effected by the equations

$$(20)\quad w_1 = \int_\infty^t \frac{dt}{\sqrt{1-t^8}}, \quad w_2 = \int_\infty^t \frac{t\,dt}{\sqrt{1-t^8}}, \quad w_3 = \int_\infty^t \frac{t^2dt}{\sqrt{1-t^8}}, \quad t^8 = \frac{z^2-1}{z^2}.$$

The substitution $t^2 = s$ reduces the first and third of these integrals to the form given in (19), and the second to an elliptic integral. This result is perfectly analogous to that of the triangle [5, 5, 2] treated by Schwarz† and Love,† and discussed here in the case for $n = 6$.

*The triangles CED and $C'E'D'$ correspond to the upper half of the fifth leaf of the z-surface.

†*Loc. cit.*

For $n = 9$, $\nu = 7$. Not any of these seven triangles can be represented conformally upon the upper half of the z-plane by integrals of deficiency 2. The triangle [6, 6, 6] is elliptic, all the others correspond to integrals of deficiency either 3 or 4.

For $n = 10$, $\nu = 8$. There are four isosceles triangles and two right triangles. The two scalene triangles [9, 8, 3] and [9, 7, 4] have associated with them the triangles [8, 6, 6] and [4, 8, 8] and correspond, respectively, to integrals of deficiency 4.

From (12) the conformal representation of the right triangles [5, 8, 7] and [5, 6, 9] is effected by the equations

$$(21) \qquad w_1 = \int_x^t \frac{dt}{\sqrt{1 - t^5}}, \quad w_2 = \int_x^t \frac{tdt}{\sqrt{1 - t^5}}, \quad 2t^5 = \frac{z - 1}{z}.$$

These integrals are, of course, exactly the same as in the case for $n = 5$, since the right triangles are, respectively, the halves of the isosceles triangles there considered. The same tetrahedron will serve for the surface which is to be simply covered.

The four isosceles triangles are [9, 9, 2], [8, 8, 4], [7, 7, 6], and [6, 6, 8]. The second and fourth of these are the ones considered in the case for $n = 5$. The first and third are also double the right triangles [5, 6, 9] and [5, 8, 7], respectively. The conformal representation of the set of four isosceles triangles is effected by the equations

$$(22) \qquad w_r = \int_x^t \frac{t^{r-1}dt}{\sqrt{1 - t^{10}}}, \quad t^{10} = \frac{z^2 - 1}{z^2}; \quad (r = 1, 2, 3, 4.)$$

The second and fourth of these integrals reduce to the form (21) by the substitution $t^2 = s$, while the first and third reduce to the same form by the substitution $t^2 = 1/s$. Thus we have a second example analogous to the triangle [5, 5, 2]. The same tetrahedron [Figure 2] can be simply covered by the triangles [9, 9, 2] and [7, 7, 6].

We have now completed our search for triangles corresponding to integrals of deficiency 2, for it is evident from formulas (9) and (12) that n cannot be greater than 12. On examination we find that not one of the twelve possible triangles that can be formed from the dodecagon, corresponds to an integral of deficiency 2 [or less] except those already discussed, as occurring in n-gons of fewer than twelve sides.

Bringing these triangles together, they are:

1.	[5, 5, 2],	whose angles are	$\pi/6$, $\pi/6$, $2\pi/3$.
2.	[4, 4, 2],	whose angles are	$\pi/5$, $\pi/5$, $3\pi/5$.
3.	[3, 3, 4],	whose angles are	$2\pi/5$, $2\pi/5$, $\pi/5$.
4.	[4, 5, 7],	whose angles are	$\pi/2$, $3\pi/8$, $\pi/8$.
5.	[5, 8, 7],	whose angles are	$\pi/2$, $\pi/5$, $3\pi/10$.
6.	[5, 6, 9],	whose angles are	$\pi/2$, $2\pi/5$, $\pi/10$.

Besides these triangles there are four others which can be reduced to the hyperelliptic case $p = 2$; viz:

1.	[7, 7, 2],	whose angles are	$\pi/8$, $\pi/8$, $3\pi/4$.
2.	[5, 5, 6],	whose angles are	$3\pi/8$, $3\pi/8$, $\pi/4$.
3.	[9, 9, 2],	whose angles are	$\pi/10$, $\pi/10$, $4\pi/5$.
4.	[7, 7, 6],	whose angles are	$3\pi/10$, $3\pi/10$, $2\pi/5$.

UNIVERSITY OF WISCONSIN,
AUGUST, 1903.

REMARKS ON A PROOF THAT A CONTINUOUS FUNCTION IS UNIFORMLY CONTINUOUS

By N. J. Lennes

1. In a note in the *Mathematische Annalen*, vol. 6 (1873), p. 319, Lüroth gave a proof* of the theorem that a function, continuous over a certain interval, is uniformly continuous over that interval. The proof is based upon the following definition of continuity at a point:

A single valued function,† $f(x)$, *is said to be continuous at a given point* $x = x_0$ *if for every positive* ϵ *there exists a positive* δ_ϵ *such that for every* x_1 *and* x_2 *on the interval* $x_0 - \delta_\epsilon \cdots x_0 + \delta_\epsilon$‡ *we have*

$$|f(x_1) - f(x_2)| < \epsilon.$$

In other words, the range§ of the function on the interval $x_0 - \delta_\epsilon \cdots x_0 + \delta_\epsilon$ is less than ϵ.

If one value of δ_ϵ can be found which satisfies the condition for the given point x_0, then clearly every smaller value, and possibly some larger ones, will satisfy the condition. Let Δ_ϵ be the largest value of δ_ϵ which satisfies the condition.‖

If the given function $f(x)$ is continuous at every point on $a \cdots b$, then for any ϵ there will be a value of Δ_ϵ corresponding to every x on $a \cdots b$. This Δ_ϵ then for any particular ϵ will be a function of x and may be denoted by $\Delta_\epsilon(x)$.

The essential part of Lüroth's proof consists in establishing the following fact: if $f(x)$ is continuous at every point of its interval, then for any particular value of ϵ, the quantity $\Delta_\epsilon(x)$ will be a continuous function of x. From this

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*Lüroth's proof is stated for functions of two variables, but applies, as he himself points out, to functions of any number of variables.

†The function $f(x)$ is supposed to be defined for every point of a given continuous interval $a \ldots b$. (We confine ourselves wholly to functions of one real variable.)

‡Or on such parts of $x_0 - \delta_\epsilon \ldots x_0 + \delta_\epsilon$ as lie within $a \ldots b$.

§ The expression "range" of a function on a given interval is used to indicate the difference between the least upper and the greatest lower bound of the function on that interval.

‖For particular values of ϵ and x_0, Δ_ϵ may clearly equal the whole distance from x_0 to "a" or "b". This is so for all values of ϵ and x_0, when $f(x)$ is a constant.

follows, by a fundamental theorem due to Weierstrass, that the function $\Delta_\epsilon(x)$ will actually reach its greatest lower bound, that is, will have a minimum value; and this minimum value, like all the other values of δ_ϵ, will be a positive quantity.

Hence: *if a single valued function is continuous over an interval $a \dots b$, then for every ϵ there exists a positive η [namely, twice the minimum value of $\Delta_\epsilon(x)$] such that the range of the function will be less than ϵ on any interval on $a \dots b$ whose length is less than η.* That is, the function is uniformly continuous and the theorem is proved.

2. A definition of continuity more familiar than that used by Lüroth is the following: *A single valued function $f(x)$ is said to be continuous at a point $x = x_0$ when for every positive ϵ there exists a positive δ'_ϵ such that for every x_1 on the interval $x_0 - \delta'_\epsilon \dots x_0 + \delta'_\epsilon$ we have*

$$|f(x_0) - (fx_1)| < \epsilon;$$

and the question arises whether this definition might have been used in place of the other in Lüroth's proof of the theorem. We may here, as before, consider the greatest δ'_ϵ which satisfies the condition, and denote it by $\Delta'_\epsilon(x)$. The question then is whether this quantity $\Delta'_\epsilon(x)$, like $\Delta_\epsilon(x)$, is a continuous function of x.

The purpose of the present note is to show that this question must be answered in the negative. The following examples prove that $\Delta'_\epsilon(x)$ is not in general a continuous function of x.

Consider the function $f(x) = \sin x$, from $x = 0$ to $x = \pi$, and take $\epsilon = \frac{1}{2}\sqrt{3}$. At $x = \frac{\pi}{3}$, $\Delta'_\epsilon(x) = \frac{2\pi}{3}$, while if x is allowed to approach $\frac{\pi}{3}$ as a limit $\left(x \text{ being greater than } \frac{\pi}{3}\right)$, $\Delta'_\epsilon(x)$ approaches $\frac{\pi}{3}$ as a limit. Hence $\Delta'_\epsilon(x)$ is discontinuous at $x = \frac{\pi}{3}$.

As another example, consider a function $f(x)$ which is represented by the following broken line. On a segment AB parallel to the X-axis there are two points C and D. On the segment CD as a base construct an isosceles triangle CED. The segments AC, CE, ED and DB, including their end points, represent a function $y = f(x)$ such that for certain values of ϵ, $\Delta'_\epsilon(x)$ is not a continuous function on the interval $A \dots B$. This case is strictly analogous to the one given above.

Instead of the one triangle DEC we now construct an infinite set of triangles as follows:

On the segment AB lay off segments BX_1, $X_1X_2 \cdots$, X_nX_{n+1}, $\cdots$ such $X_nX_{n+1} = \dfrac{X_{n-1}X_n}{2}$. On segment BX_1, H_1 is any point. K_n is a point on X_nX_{n+1} such that $X_nK_n = \dfrac{X_nX_{n+1}}{3}$; $H_n (n \geqq 2)$ is a point on $X_{n-1}X_n$ such that $H_nX_n = \dfrac{X_{n-1}X_n}{3}$.

On the H_nK_n as bases construct isosceles triangles with vertices V_n and altitudes h_n such that $h_n = \frac{2}{3}(h_{n-1})$.

The segments BH_1, H_1V_1, V_1K_1, K_1H_2, $\cdots$, H_nV_n, V_nK_n, K_nH_{n+1}, $\cdots$ including their end points represent a function $f(x)$ such that for *every* value of ϵ which is less than the range of the function on its interval, there is a point at which $\Delta'_\epsilon(x)$ is a discontinuous function of x.

The function $y = x \sin \dfrac{1}{x}$ is such that for some values of ϵ within an interval $0 \cdots \Delta$ [Δ being positive and not zero], $\Delta_\epsilon(x)$ is discontinuous at some point no matter how small Δ is. This is also true for ϵ on an interval $\Delta \cdots + \infty$ no matter how large Δ is.

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